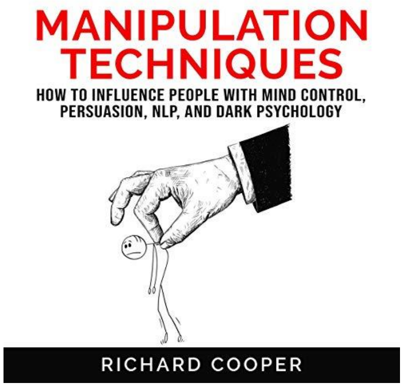
3. TECHNIQUES OF INTEGRATION

§3.1. Integrating Functions In Terms of Elementary Functions



While there are efficient techniques for calculating definite integrals to any desired degree of accuracy it's often useful to find an indefinite integral, as an explicit function. With differentiation, any function we might write down in terms of polynomials, trigonometric, logarithmic and exponential functions can be differentiated explicitly using the Sum, Product, Quotient and Chain Rules. Things are not so easy for integration. There are many functions, such as e^{x^2}, where the indefinite integral, or anti-derivative, can't be expressed in terms of the above elementary functions. We're forced, in such cases, to invent new functions.

When you first met integration you were only able to integrate a function by noticing that something like it had arisen as the derivative of some function. By a little reverse engineering you were able to find the integral. Here we shall develop some techniques for finding some harder integrals. The simplest of these techniques is integration by substitution.

§3.2. Integration By Substitution

Theorem 1: If $u = \mathrm{g}(x)$ and this function is differentiable then

$$\int \mathrm{f}(x)\,\mathrm{d}x = \int \frac{\mathrm{f}(x)}{\mathrm{g}'(x)}\mathrm{d}u\ .$$

Proof: Let $y = \displaystyle\int \frac{\mathrm{f}(x)}{\mathrm{g}'(x)}\mathrm{d}u$.

Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}$ by the Chain Rule for differentiation

$$= \frac{\mathrm{f}(x)}{\mathrm{g}'(x)}\,.\mathrm{g}'(x)$$

$$= \mathrm{f}(x).$$

Hence, by the Fundamental Theorem of Calculus,

$$y = \int \mathrm{f}(x)\mathrm{d}x\ .$$

Of course we need to be able to express $\dfrac{\mathrm{f}(x)}{\mathrm{g}'(x)}$ entirely in terms of u and then we have to be able to integrate this function of u.

Example 1: Evaluate $\int x e^{(x^2 + 3)} dx$.

Solution: Here $f(x) = xe^{(x^2 + 3)}$. Let $u = g(x) = x^2 + 3$.

Then $g'(x) = 2x$ and $\dfrac{f(x)}{g'(x)} = ½\, e^{(x^2 + 3)} = ½\, e^u$.

$$\therefore \int x e^{(x^2 + 3)} dx = \frac{1}{2}\int e^u du$$

$$= \frac{1}{2} e^u + c$$

$$= \frac{1}{2}\, e^{(x^2 + 3)} + c.$$

NOTES: (1) It is essential that you don't leave your answer in terms of u. If it is an integral with respect to x you must express the integral in terms of x.

(2) Other substitutions would have done just as well. We could have let $u = x^2$ and obtained:

$$\int x e^{(x^2 + 3)} dx = \frac{1}{2}\int e^{u+3} du$$

$$= \frac{1}{2} e^3 \int e^u du$$

$$= \frac{1}{2} e^{u+3} + c$$

$$= \frac{1}{2}\, e^{(x^2 + 3)} + c.$$

(3) With definite integrals it is not necessary to express your answer in terms of x, provided to update your limits to the corresponding values of u. After all, with a definite integral the answer is always a number. So with the substitution $u = x^2 + 3$:

$$\int_1^2 xe^{(x^2+3)}\,dx = \frac{1}{2}\int_4^7 e^u du) = ½\ \left[e^u\right]_4^7 = ½\ (e^7 - e^4)$$

(4) It's not always easy to remember the expression $\int \frac{f(x)}{g'(x)} du$. That's why we make use of the wonderfully suggestive notation of $\frac{du}{dx}$ and pretend that it's a fraction rather than the limit of a fraction. So to solve $\int xe^{(x^2+3)}dx$ we put $u = x^2 + 3$. Then $\frac{du}{dx} = 2x$.

We now write du = 2x dx. So we replace dx in the integral by $\frac{du}{2x}$ to get $\int xe^u \frac{du}{2x} = \int e^u \frac{du}{2}$ etc.

Now to separate $\frac{du}{dx}$ as if it is a fraction is pure nonsense and in no way is it logically justified. Yet the beauty of the Leibniz notation is that it works! But just in case you have a particularly pedantic lecturer it would be best to leave out the steps where the du and dx float around as separate entities. So, no lecturer would object to the following:

Example 1 (simplified notation):

Putting $u = x^2 + 3$ we get $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$.

So $\int x\mathrm{e}^{(x^2+3)}\mathrm{d}x = \int x\mathrm{e}^{u} \frac{\mathrm{d}u}{2x}$

$$= \int \mathrm{e}^{u} \frac{\mathrm{d}u}{2}$$

$$= \tfrac{1}{2}\,\mathrm{e}^{u} + \mathrm{c} = \tfrac{1}{2}\,\mathrm{e}^{(x^2+3)} + c.$$

You may wonder why nobody objects to writing $\frac{\mathrm{d}u}{2}$ as if du has a life of its own. But du and dx are considered acceptable within the context of an integral. Just don't let du and dx appear anywhere other than in the $\frac{\mathrm{d}u}{\mathrm{d}x}$ notation or an integral.

All the wonderful things we can do with the Leibniz notation can be justified logically. Remember the Chain Rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$? We pretend that we're cancelling the du's, which is sheer nonsense. But we can justify it by using the fact that the limit of a product is the product of the limits. Every time you do something like this, raise your glass to Leibniz for inventing such wonderful notation!

Example 2: Find $\int_0^{\pi/3} \tan x \, dx$.

Solution: Here the substitution is not as obvious as in the previous example.

Write $\int_0^{\pi/3} \tan x \, dx = \int_0^{\pi/3} \frac{\sin x}{\cos x} \, dx$.

Let's try $u = \cos x$. When $x = 0$, $u = 1$ and when $x = \pi/3$, $u = ½$.

Then $du = -\sin x \, dx$, so $dx = \frac{du}{-\sin x}$ (Once again you'd better never write such things. I don't object to it because I interpret this as "in the context of an integral, dx can be replaced by $\frac{du}{-\sin x}$". To be on the safe side you'd better leave out this step and go directly to the next line.)

So $\int_0^{\pi/3} \tan x \, dx = \int_0^{\pi/3} \frac{\sin x}{\cos x} \, dx$

$$= \int_1^{½} \frac{\sin x}{u} \frac{du}{(-\sin x)}$$

$$= -\int_{1}^{1/2} \frac{1}{u}\,\mathrm{d}u = \int_{1/2}^{1} \frac{1}{u}\,\mathrm{d}u$$

$$= [\log u]_{1/2}^{1} = 0 - (-\log 2) = \log 2.$$

As an exercise you could try the alternative substitution $u = \sin x$. This will work, though you'll find it not quite as easy. Whenever you are thinking of a substitution remember that you are going to "divide by the derivative". This principle makes the substitution $u = \cos x$ much more attractive than $u = \sin x$.

Example 3: Find $\int e^{x^2}\mathrm{d}x$.

Solution: Let $u = x^2$.

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, so $\mathrm{d}u = 2x\,\mathrm{d}x$ and hence $\mathrm{d}x = \frac{\mathrm{d}u}{2x}$.

$\therefore \int e^{x^2}\mathrm{d}x = \int e^{u}\frac{\mathrm{d}u}{x}$.

We need to express the integrand entirely in terms of u. This can be done.

$\int e^{x^2}\mathrm{d}x = \int \frac{e^{u}}{\sqrt{u}}\,\mathrm{d}u$. But now we're stuck. We might think of making a further substitution $v = \sqrt{u}$ but then we would be back to the original integral. And this, as I've said, cannot be found, at least not in terms of the elementary functions that we know about.

Example 4: Find $\int \sin(\log x)\, \mathrm{d}x$.

Solution: Let $u = \log x$.

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ and so $\mathrm{d}x = x\mathrm{d}u = \mathrm{e}^u\, \mathrm{d}u$.

So $\int \sin(\log x)\, \mathrm{d}x = \int \mathrm{e}^u \sin u\, \mathrm{d}u$.

That's all very well, but how do we go about integrating $\mathrm{e}^u \sin u$? This one can be done but it needs another technique, one that we'll discuss in the next section.

§3.3. Integration by Parts

For integration we don't have a product rule like we do for differentiation. Integration by parts is a technique that reverses the product rule.

Theorem 2 (Integration by Parts):
Let u and v be functions of x.

$$\text{Then } \int u\, \mathrm{d}v = uv - \int v\, \mathrm{d}u .$$

Proof: Since $\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$, it follows that

$uv = \int u\, \mathrm{d}v + \int v\, \mathrm{d}u$ ☺

Example 5: Find $\int x \cos x \, dx$.

Solution: Let $u = x$ and $dv = \cos x \, dx$. (The latter is equivalent to letting $\frac{dv}{dx} = \cos x$.)

Since $u = x$, $du = dx$ (not by multiplying both sides by d – that would be ridiculous – but by observing that $\frac{du}{dx} = \frac{dx}{dx} = 1$, Again the latter doesn't result from dividing top and bottom by d*x* but by remembering that the derivative of *x*, with respect to *x*, is 1, But see how suggestive the Leibniz notation is!

From $dv = \cos x \, dx$ we get $v = \int \cos x \, dx = \sin x$.

So, integrating by parts:

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c.$$

With an Integration by Parts we have *two* integrations steps. The first is where we integrate d*v* to get *v* and the second is where we find $\int v \, du$. With the first integration we don't need to include an arbitrary constant. Any indefinite integral of d*v* will do. It's only with the second integration that we need to include the arbitrary constant. (If we did include an arbitrary constant with the first integration it would disappear at the end.)

Example 6 (with an unnecessary extra arbitrary constant): Find $\int x \cos x \, dx$.

Solution: Let $u = x$ and $dv = \cos x \, dx$.

Then $du = d\underline{x}$, meaning that $\frac{du}{dx} = 1$, and

$v = \int \cos x \, dx = \sin x + c_1$.

So $\int x \cos x \, dx = x \sin x + c_1 x - \int (\sin x + c_1) \, dx$

$= x \sin x + c_1 x + \cos x - c_1 x + c$.
$= x \sin x + \cos x + c$

Example 7: Find $\int e^x \sin x \, dx$.

Solution: Let $u = e^x$ and $dv = \sin x \, dx$.

Then $du = e^x dx$ and $v = -\cos x$.

Hence $\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$.

Looks like we need a second integration by parts.
Now let $u = e^x$ and $dv = \cos x \, dx$.

Then $du = e^x dx$ and $v = \sin x$.

Then $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$.

It looks like we're going around in circles. To find $\int e^x \sin x \, dx$ we need to be able to find $\int e^x \sin x \, dx$!

But wait! Substituting into the first integration by parts we get:

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$.

Hence $2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$ and so

$\int e^x \sin x \, dx = \frac{1}{2} e^x(\sin x - \cos x) + c.$

The moral of the story is that often we appear to be going around in circles and yet there is a subtle difference that can be useful.

I remember driving around in a multi-storey car park with my mother-in-law. We'd been around several times without finding a parking place. By now we were on the fifth floor. She said, "why do you keep going around in circles – you can see that all the places are full!"

We were indeed going around in circles but we were going to higher and higher levels.

In the above example, we got back to where we started after two applications of integration by parts but with one important difference – a change of sign.

With $e^x \sin x$ we could have associated the e^x term with either u or dv. It would not have mattered which we chose. The same choice applies at the second stage. But it's important that we don't switch. If we let $u = e^x$ we'd better do this at the second stage as well. If we changed we'd indeed end up exactly where we started. Try it.

Example 8: Find $\int x^2 e^x \, dx$.

Solution: Let $u = x^2$ and $dv = e^x \, dx$.

Then $du = 2x \, dx$ and $v = e^x$.

So $\int x^2 e^x \, dx = x^2 e^x - 2\int x \, e^x \, dx$.

Here we've expressed our integral in terms of another one that we can't find directly.
But one more integration by parts will finish it off.

$\boldsymbol{\int x \, e^x \, dx}$ **:**

Let $u = x$ and $dv = e^x \, dx$. (We're recycling the symbols u, v since we no longer need their original values.)

Then $du = dx$ and $v = e^x$. (Note: $du = dx$ because $\dfrac{du}{dx} = 1$.)

Hence $\int x\, e^x\, dx = x\, e^x - \int e^x\, dx$

$$= x\, e^x - e^x.$$

Substituting we get

$$\int x^2\, e^x\, dx = x^2\, e^x - 2(x\, e^x - e^x) + c$$

$$= (x^2 - 2x + 1)\, e^x + c.$$

Integration by parts can be tried whenever we have a product, but there's the question of which factor should be the 'u' and which should go into the 'dv'. Let's see what happens in the above integral if we reverse the roles of the factors.

Example 8 (a less successful attempt): Find $\int x^2\, e^x\, dx$.

Solution: Let $u = e^x$ and $dv = x^2\, dx$.

Then $du = e^x$ and $v = 1/3\, x^3$.

So $\int x^2\, e^x\, dx = ½\, x^2\, e^x - 1/3 \int x^3\, e^x\, dx$.

Clearly we've made things worse.

Some thought needs to be given as to what part of the integral we treat as the 'u' and what part becomes the 'dv'. Notice that the 'u' gets differentiated while the 'dv' gets integrated. If there's a power of x then usually this should be the u, because the power will come down by one. But there *are* cases where the power of x should go with the dv. This is when the other factor is a log.

The following table gives an order of priority for what should be the u.

logs must be differentiated
powers of x should be differentiated if there are no logs
sines and cosine and exponentials can be differentiated if there are no logs or powers of x

In a few cases integration by parts can be used where there's no obvious product. In such cases we can try writing $\int f(x)\,dx$ as $\int 1.f(x)\,dx$.

Clearly we don't try $u = x$ and $dv = f(x)dx$. Try it. You will end up with $\int f(x)\,dx = \int f(x)\,dx$ which is of no use.

Example 9: Find $\int \log x\,dx$.

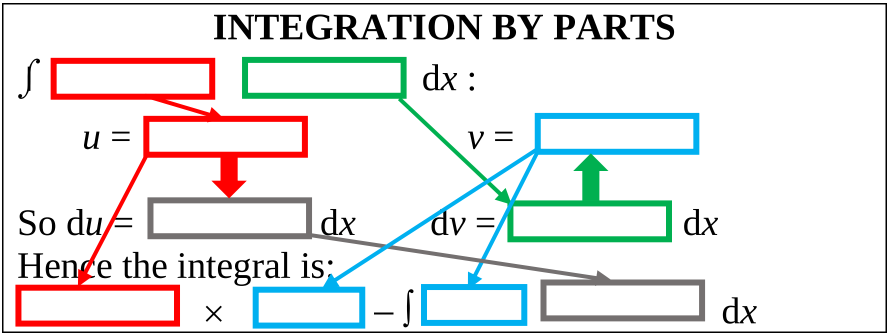
Solution: Let $u = \log x$ and $dv = dx$.

Then $du = \frac{dx}{x}$ $\left(\text{Since } \frac{du}{dx} = \frac{1}{x}\right)$ and $v = x$.

So $\int \log x\,dx = x \log x - \int dx$

$= x \log x - x + c$.

My suggested layout for an Integration by Parts problem is as follows:



In some cases, where we have an integral of the form $\text{I}_n = \int x^n \, \text{f}(x)\text{d}x$ we can obtain an explicit expression for it in terms of n. We generally do this by using integration by parts to express I_n in terms of I_{n-1} to obtain what is called a **reduction formula**. We'll explore this process in a later section. But in the following case it can be done all in one go.

Example 10: Find $\int x^n \log x \, \text{d}x$.

Solution: Let $u = \log x$ and $\text{d}v = x^n \, \text{d}x$.

Then $\text{d}u = \dfrac{1}{x}$ and $v = \dfrac{1}{n+1} x^{n+1}$.

$$\text{So } \int x^n \log x \, dx = \frac{1}{n+1} x^{n+1} \log x - \frac{1}{n+1} \int x^n \, dx$$
$$= \frac{1}{n+1} x^{n+1} \log x - \frac{1}{(n+1)^2} x^{n+1}$$
$$= \frac{x^{n+1}}{(n+1)^2} [(n+1)\log x - 1].$$

§3.4. Trigonometric Substitution

There are two substitutions, involving inverse trigonometric functions, that are very useful.

They are $\theta = \sin^{-1}x$ and $\theta = \tan^{-1}x$ or, to quote them in the more natural direction:

$$x = \sin\theta \text{ and } x = \tan\theta.$$

The derivatives are $\cos\theta$ and $\sec^2\theta$ respectively.

When the integrand involves $1 + x^2$:

Put $x = \tan\theta$;
$1 + x^2 = \sec^2\theta$;
$dx = \sec^2\theta \, d\theta$.

When the integrand involves $1 - x^2$:

Put $x = \sin\theta$;
$1 - x^2 = \cos^2\theta$;
$dx = \cos\theta \, d\theta$.

Example 11: Find $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$.

Solution: Let $x = \tan\theta$.

Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$.

So $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \int_0^{\pi/4} \frac{1}{\sec^2\theta}.\sec^2\theta\,\mathrm{d}\theta$

$= \int_0^{\pi/4} \mathrm{d}\theta$ (This is the integral of 1.)

$= [\theta]_1^{\pi/4}$

$= \pi/4.$

Example 12: Find $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$.

Solution: Let $x = \sin\theta$.

Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$.

So $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{\cos\theta}.\cos\theta\mathrm{d}\theta$

$= \int \mathrm{d}\theta$

$= \theta + c$

$= \sin^{-1}x + c.$

This technique can be easily adapted to case that involve other quadratic expressions. For example if our integral involves $9 + x^2$ we'd let $x = 3\tan\theta$.

Then $dx = 3\sec^2\theta$ and $9 + x^2 = 9(1 + \tan^2\theta) = 9\sec^2\theta$.

Example 13: Find $\int \frac{x}{9 + x^2} dx$.

Solution: Let $x = 3\tan\theta$.

Then $dx = 3\sec^2\theta \, d\theta$.

So $\int \frac{x}{9 + x^2} dx = \int \frac{3\tan\theta}{9\sec^2\theta} . 3\sec^2\theta \, d\theta$

$$= \int \tan\theta \, d\theta$$

$$= -\log\cos\theta + c \text{ (from Example 3)}$$

$$= -\log\cos(\tan^{-1}(x/3)) + c$$

$$= -\log\left(\frac{3}{\sqrt{x^2 + 9}}\right) + c$$

$$= \log\left(\frac{\sqrt{x^2 + 9}}{3}\right) + c.$$

Trigonometric substitutions can be used whenever the integral involves a quadratic expression, using the technique of completing the square. The quadratic

$$x^2 + bx + c$$

can be written as:

$$(x + ½\, b)^2 + (c - ¼\, b^2).$$

If $c \geq \frac{1}{4} b^2$ we let $u = x + \frac{1}{2} b$ and $a = \sqrt{c - \frac{1}{4} b^2}$ to get:

$$u^2 + a^2.$$

Then we try the substitution $u = a.\tan\theta$

If $c < \frac{1}{4} b^2$ we let $u = x + \frac{1}{2} b$ and $a = \sqrt{\frac{1}{4} b^2 - c}$ to get:

$$u^2 - a^2.$$

Then we try the substitution $u = a.\sin\theta$.

Example 14: Find $\displaystyle\int_{-1}^{1} \frac{1}{x^2 + 2x + 5}\,dx$.

Solution:

$x^2 + 2x + 5 = (x + 1)^2 + 4$. Let $u = x + 1$ and $a = 2$.

When $x = -1$, $u = 0$ and when $x = 1$, $u = 2$.

Therefore $\displaystyle\int_{-1}^{1} \frac{1}{x^2 + 2x + 5}\,dx = \int_{0}^{2} \frac{1}{u^2 + 4}\,du$.

Now let $u = 2\tan\theta$. Then $du = 2\sec^2\theta\, d\theta$.

When $u = 0$, $\theta = 0$ and when $u = 2$, $\theta = \pi/4$.

Hence $\displaystyle\int_{0}^{2} \frac{1}{u^2 + 4}\,du = \int_{0}^{\pi/4} \frac{2\sec^2\theta}{2\sec^2\theta}\,d\theta = \int_{0}^{\pi/4} d\theta = [\theta]_{0}^{\pi/4} = \pi/4$.

§3.5. Partial Fractions

A rational function is one that can be expressed as a polynomial divided by a polynomial, such as

$$f(x) = \frac{x^5 + 4x^4 - 12x^3 - 11x^2 + 27x - 3}{x^3 + 4x^2 - 9x + 1}.$$

If the denominator has lower degree than the numerator, as in this case, we can carry out long division to get a polynomial where the degree of the denominator is larger than that of the numerator.

In this case we divide

$x^3 + 4x^2 - 9x + 1$ into

$x^5 + 4x^4 - 12x^3 - 11x^2 + 27x - 3$ to get

$$f(x) = x^2 - 3 + \frac{x^2 + 5x - 7}{x^3 + 4x^2 - 9x + 1}.$$

The polynomial bit is no problem when it comes to integration so we now focus on rational functions where the denominator has higher degree than the numerator.

Theorem 3: Every real polynomial factorises into linear and quadratic factors with real coefficients.

Proof: If α is a real zero then $x - \alpha$ is a real linear factor.

If α is a non real zero then so is its conjugate $\overline{\alpha}$.

Then $(x - \alpha)(x - \overline{\alpha}) = x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha}$ is a factor.

But $\alpha + \bar{\alpha} = 2\text{Re}(\alpha)$ and $\alpha\bar{\alpha} = |\alpha|^2$ are both real so we have a real quadratic factor. ✋☺

Example 15: Let $f(x) = x^5 - x^4 - 3x^2 - x - 2$.
Then $f(2) = 0$ so $x - 2$ is a factor. The other factor is:

$$x^4 + x^3 + 2x^2 + x + 1 = (x^2 + x + 1)(x^2 + 1).$$

Theorem 4: If $f(x) = \dfrac{a(x)}{b(x)}$ where $a(x)$ and $b(x)$ are real polynomials with $\deg a(x) < \deg b(x)$ then $f(x)$ can be expressed as a sum of rational functions of the form

$$\frac{k}{(x-b)^n} \text{ and of the form}$$

$$\frac{kx+h}{(x^2+ex+f)^n}$$

where each n is a positive integer, the coefficients are real and the quadratic denominators have non-real zeros.
Proof: We don't offer a proof here as it is rather long and tedious. In each specific instance it can be demonstrated to be true. Those who really wish to see a proof will find one in the appendices.

The method for splitting a rational function into partial fractions is as follows:

(1) Divide the numerator by the denominator to obtain a polynomial plus a rational function where the degree of the numerator is less than that of the denominator.

(2) Factorise the denominator into real factors of degree 1 or 2. We assume that any quadratic factor that can be factorised further, over the reals, has already been so factorised.

(3) Equate the rational function into a sum of terms of the form $\dfrac{A}{(ax+b)^r}$ and $\dfrac{Bx + C}{(ax^2 + bx + c)^r}$. Include such terms for every r for which such an $(ax + b)^r$ or $(ax^2 + bx + c)^r$ divides the denominator.

(4) Multiply each side of this equation by the factorised denominator. This will result in an equation in which the numerator is said to be identically equal to a sum of factorised polynomials.

When I first learnt about partial fractions I was taught to use the symbol ≡ instead of the usual = sign. The reason for this is to emphasise that the polynomials are equal as polynomials, lest we be tempted to try to solve it for x. I can’t see any good reasons for insisting on this. We can surely distinguish between an equality that holds for all x, or certain x. However old habits die hard and I can’t resist on continuing to use ≡ in this context.

(5) The above expression will have symbols representing real constants. These we must now find.

There are two techniques for doing this.
(A) Substitute values for *x*. Each such value will give a linear relation between the said constants. Accumulating enough of these equations we can find the constants.

Now since we are free to choose these values of x, we choose such values as will simplify these equations.

(B) Equate corresponding coefficients of powers of x. This will give further linear equations that are available for use.

In general a combination of these methods is used so as to simplify the task in finding the constants.

1	**DIVIDE**
2	**FACTORISE**
3	**EQUATE**
4	**MULTIPLY**
5	**FIND CONSTANTS**

Example 16: Separate $\dfrac{x^2 + 2x + 1}{(x + 3)(x - 1)(x - 2)}$ into partial fractions.

Solution:

(1) The numerator has smaller degree than the denominator, so step (1) is not required here.

(2) Here the denominator is already factorised.

(3) Put $\dfrac{x^2 + 2x + 1}{(x + 3)(x - 1)(x - 2)} \equiv \dfrac{A}{x + 3} + \dfrac{B}{x - 1} + \dfrac{C}{x - 2}$

(4) Hence $x^2 + 2x + 1 \equiv A(x - 1)(x - 2) + B(x + 3)(x - 2) + C(x + 3)(x - 1)$

Notice that we don't bother to do this. The only reason why we might want to do this is in the case where the numerator and denominator have a common factor. If we happen to notice this then by all means do the cancellation. But if you don't happen to notice a common factor you will just end up with a zero constant.

(5) Let $x = 1$. Then $4 = -4B$, so $B = -\dfrac{1}{4}$.

Let $x = 2$. Then $9 = 5C$, so $C = \dfrac{9}{5}$.

Let $x = -3$. Then $4 = 20A$, so $A = \dfrac{1}{5}$.

So $\dfrac{x^2 + 2x + 1}{(x + 3)(x - 1)(x - 2)} \equiv \dfrac{1/5}{x + 3} - \dfrac{1/4}{x - 1} + \dfrac{9/5}{x - 2}$

We only needed to use technique (A) and, by judicious choices, the job of solving the equations was easy.

Example 17: Separate $\frac{x^4 + x^3 + 1}{x^2 - 6x + 5}$ into partial fractions.

Solution:

(1) Dividing $x^2 - 6x + 5$ into $x^4 + x^3 + 1$ we find that $\frac{x^4 + x^3 + 1}{x^2 - 6x + 5} \equiv x^2 + 7x + 37 + \frac{187x - 184}{x^2 - 6x + 5}$.

Note that we can check this by putting $x = 0$.

(2) Factorising $x^2 - 6x + 5$ we get $(x - 1)(x - 5)$.

(3) $\frac{187x - 184}{x^2 - 6x + 5} \equiv \frac{187x - 184}{(x - 1)(x - 5)}$

$$\equiv \frac{A}{x - 1} + \frac{B}{x - 5}.$$

(4) $\therefore$ 187x – 184 ≡ A(x – 5) + B(x – 1).

(5) Put x = 1. Then 3 = – 4A and so A = $-\frac{3}{4}$.

Put x = 5. Then 751 = 4B and so B = $\frac{751}{4}$.

Hence $\frac{x^4 + x^3 + 1}{x^2 - 6x + 5} \equiv x^2 + 7x + 37 - \frac{3/4}{x - 1} + \frac{751/4}{x - 5}$.

Example 18: Express $\dfrac{x^2 - x - 6}{x^3 - x^2 - x + 1}$ as partial fractions.

Solution:

(1) Not needed.

(2) $x^3 - x^2 - x + 1 \equiv (x - 1)(x^2 - 1) \equiv (x - 1)^2(x + 1)$.

(3) Let $\dfrac{x^2 - x - 6}{x^3 - x^2 - x + 1} \equiv \dfrac{A}{(x - 1)^2} + \dfrac{B}{x - 1} + \dfrac{C}{x + 1}$.

(4) $\therefore\ x^2 - x - 6 \equiv A(x + 1) + B(x - 1)(x + 1) + C(x - 1)^2$.

(5) Put x = 1. Then – 6 = 2A, so A = – 3.

Put x = – 1. Then – 4 = 4C, so C = – 1.

Equate coefficients of x^2, 1 = B + C. Hence B = 2.

Example 19: Express $\dfrac{x^2 + 3}{(x^2 + x + 1)(x - 2)}$ in Partial Fractions.

(1) and (2) are not required.

(3) Let $\dfrac{x^2 + 3}{(x^2 + x + 1)(x - 2)} \equiv \dfrac{Ax + B}{x^2 + x + 1} + \dfrac{C}{x - 2}$.

(4) $\therefore\ x^2 + 3 \equiv (Ax + B)(x - 2) + C(x^2 + x + 1)$.

(5) Put $x = 2$. Then 7 = 7C and so C = 1.

Equate coefficients of x^2: 1 = A + C, so A = 0.

Equate constant terms (or put x = 0): 3 = – 2B + C, so B = – 1.

Hence $\dfrac{x^2 + 3}{(x^2 + x + 1)(x - 2)} \equiv -\dfrac{1}{x^2 + x + 1} + \dfrac{1}{x - 2}$.

Example 20: Express $\dfrac{2x^5 + 3x^2 - 6x + 3}{(x^2+1)^2(x-1)^2}$ in terms of Partial Fractions.

Solution:

(1) and (2) Not required.

(3) Let $\dfrac{2x^5 + 3x^2 - 6x + 3}{(x^2+1)^2(x-1)^2}$

$$\equiv \frac{A}{(x^2+1)^2} + \frac{B}{x^2+1} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

(4) $\therefore\ 2x^5 + 3x^2 - 6x + 3 \equiv A(x-1)^2 + B(x^2+1)(x-1)^2$
$+ C(x^2+1)^2 + D(x-1)(x^2+1)^2.$

(5) Let $x = 1$: $2 = 2C$, so $C = 1$.

Equate coefficients of x^5: $2 = D$.

Equate coefficients of x^4: $0 = B + C - D$, so $B = 1$.

Equate constant terms: $3 = A + B + C - D$, so $A = 3$.

Hence $\dfrac{2x^5 + 3x^2 - 6x + 3}{(x^2+1)^2(x-1)^2}$

$$\equiv \frac{3}{(x^2+1)^2} + \frac{1}{x^2+1} + \frac{1}{(x-1)^2} + \frac{2}{x-1}.$$

The consequence of Theorem 4, with regard to integrating rational functions, is that we can integrate any rational function, provided we can factorise the denominator and provided we can integrate functions of the form $\dfrac{1}{(x-b)^n}$ and $\dfrac{x+h}{(x^2+ex+f)^n}$. To integrate

$\dfrac{1}{(x-b)^n}$ we make the substitution $u = x - b$. If $n \geq 2$ the resulting integral will be a constant multiple of a function of the same form.
If $n = 1$ it will be $\log(x - b)$.

A term of the form $\dfrac{x+h}{(x^2+ex+f)^n}$ can be written as:

$$\frac{\tfrac{1}{2}(2x+e)}{(x^2+ex+f)^n} + \frac{h - \tfrac{1}{2}e}{(x^2+ex+f)^n} \; .$$

Since $2x + e$ is the derivative of $x^2 + ex + f$, the first term can be integrated by the substitution $u = x^2 + ex + f$, giving a constant multiple of a function of the same form if $n \geq 2$ and $\tfrac{1}{2}\log(x^2 + ex + f)$ if $n = 1$.
It remains to discuss the integration of $\dfrac{1}{(x^2+ex+f)^n}$.

The first step is to complete the square, that is write

$$x^2 + ex + f = (x + \tfrac{1}{2}\mathrm{e})^2 + (f - \tfrac{1}{4}e^2).$$

Because $x^2 + ex + f$ has no real zeros, $f - \tfrac{1}{4}e^2 > 0$ and so we may write it as a^2 for some real number, a.
By substituting $u = \dfrac{x + \tfrac{1}{2}e}{a}$ we can express $\dfrac{1}{(x^2+ex+f)^n}$ in the form $\dfrac{1}{(u^2+1)^n}$.

Now, what is $\int \frac{1}{(u^2+1)^n}\,du$?

Make the trigonometric substitution $u = \tan\theta$.

Since $du = \sec^2\theta\, d\theta$ and $u^2 + 1 = \sec^2\theta$,

$$\int \frac{1}{(u^2+1)^n}\,du = \int \frac{1}{\sec^{2n-2}\theta}d\theta = \int \cos^{2n-2}\theta\, d\theta .$$

If $n = 1$ this is just $\int d\theta = \theta = \tan^{-1}u$.

For $n \geq 2$ we need to do some further work.

§3.6. Reduction Formulae

Theorem 5: If $n \geq 2$ and $I_n = \int \cos^n\theta\, d\theta$ then

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \cos^{n-1}\theta.\sin\theta.$$

Proof: Let $u = \cos^{n-1}\theta$ and $dv = \cos\theta\, d\theta$.

Then $du = -(n-1)\cos^{n-2}\theta.\sin\theta$ and $v = \sin\theta$.

Hence $I_n = \cos^{n-1}\theta.\sin\theta + (n-1)\int \cos^{n-2}\theta.\sin^2\theta\, d\theta$

$$= \cos^{n-1}\theta.\sin\theta + (n-1)\int \cos^{n-2}\theta.(1-\cos^2\theta)\, d\theta$$

$$= \cos^{n-1}\theta.\sin\theta + (n-1)(I_{n-2} - I_n).$$

So $nI_n = (n-1)I_{n-2} + \cos^{n-1}\theta.\sin\theta$

and therefore $I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \left(\frac{1}{n}\right)\cos^{n-1}\theta.\sin\theta.$

If n is even we can use this Reduction Formula until we eventually get down to I_0.

Clearly $I_0 = \int d\theta = \theta + c$.

If n is odd we can express I_n in terms of

$$I_1 = \int \cos\theta \, d\theta = \sin\theta + c.$$

Example 21: Find $\int \cos^4\theta \, d\theta$.

Solution: $I_4 = \left(\frac{3}{4}\right) I_2 + \left(\frac{1}{4}\right) \cos^3\theta.\sin\theta.$

$$I_2 = \left(\frac{1}{2}\right) I_0 + \left(\frac{1}{2}\right) \cos\theta.\sin\theta.$$

Hence $I_4 = \left(\frac{3}{4}\right) \left(\left(\frac{1}{2}\right) I_0 + \frac{1}{2}\cos\theta.\sin\theta\right) + \left(\frac{1}{4}\right) \cos^3\theta.\sin\theta$

$$= \frac{3}{8}\theta + \frac{3}{8}\cos\theta.\sin\theta + \frac{1}{4}\cos^3\theta.\sin\theta.$$

Example 22: Find $\int \frac{2x + 3}{(x^2 + 2x + 5)^3} \, dx$.

Solution: I will omit the arbitrary constants until the very end.

$\int \frac{2x + 3}{(x^2 + 2x + 5)^3} \, dx$

$$= \int \frac{2x + 2}{(x^2 + 2x + 5)^3} \, dx + \int \frac{1}{(x^2 + 2x + 5)^3} \, dx .$$

$\displaystyle\int\frac{\mathbf{2x+2}}{\mathbf{(x^2+2x+5)^3}}\,\mathbf{dx}$ **:** Let $u = x^2 + 2x + 5$.

Then $du = (2x + 2)dx$.

So this integral is $\displaystyle\int\frac{du}{u^3} = -\tfrac{1}{2}u^{-2}$

$$= -\frac{1}{2(x^2+2x+5)^2}\,.$$

$\displaystyle\int\frac{\mathbf{1}}{\mathbf{(x^2+2x+5)^3}}\,\mathbf{dx}$ **:** Write $x^2 + 2x + 5$ as $(x + 1)^2 + 4$.

Let $u = \dfrac{x+1}{2}$. Then $du = \tfrac{1}{2}\,dx$, that is, $dx = 2du$ and

$$x^2 + 2x + 5 = 4(1 + u^2).$$

This integral is $\displaystyle\frac{2}{4^3}\int\frac{1}{(1+u^2)^3}du = \frac{1}{32}\int\frac{1}{(1+u^2)^3}du$.

$\displaystyle\int\frac{\mathbf{1}}{\mathbf{(1+u^2)^3}}\mathbf{du}$ **:** Let $u = \tan\theta$.

Then $du = \sec^2\theta\, d\theta =$ and $1 + u^2 = \sec^2\theta$.

So this integral is $\displaystyle\int\frac{1}{\sec^4\theta}d\theta = \int\cos^4\theta\, d\theta$.

From Example 21 this is:

$$\frac{3}{8}\theta + \frac{3}{8}\cos\theta.\sin\theta + \frac{1}{4}\cos^3\theta.\sin\theta.$$

Since $u = \tan\theta$, $\sin\theta = \dfrac{u}{\sqrt{1+u^2}}$ and $\cos\theta = \dfrac{1}{\sqrt{1+u^2}}$.

So $\displaystyle\int \frac{1}{(1+u^2)^3}\,\mathrm{d}u = \frac{3}{8}\tan^{-1}u + \frac{3}{8}\left(\frac{u}{1+u^2}\right) + \frac{1}{4}\left(\frac{u}{(1+u^2)^2}\right).$

Since $u = \dfrac{x+1}{2}$ we can write this as:

$$\frac{3}{8}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{3}{8}\left(\frac{2x+2}{x^2+2x+5}\right) + \frac{1}{4}\left(\frac{8x+8}{(x^2+2x+5)^2}\right).$$

$$= \frac{3}{8}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{3}{4}\left(\frac{x+1}{x^2+2x+5}\right) + \left(\frac{2x+2}{(x^2+2x+5)^2}\right).$$

Hence $\displaystyle\int \frac{1}{(x^2+2x+5)^3}\,\mathrm{d}x = \frac{1}{32}\int \frac{1}{(1+u^2)^3}\mathrm{d}u$

$$= \frac{3}{256}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{3}{128}\left(\frac{x+1}{x^2+2x+5}\right) + \frac{1}{16}\left(\frac{x+1}{(x^2+2x+5)^2}\right).$$

Finally we add $\displaystyle\int \frac{2x+2}{(x^2+2x+5)^3}\,\mathrm{d}x$

$$= -\frac{1}{2(x^2+2x+5)^2} = -\frac{8}{16}\left(\frac{1}{(x^2+2x+5)^2}\right).$$

Hence $\displaystyle\int \frac{2x+3}{(x^2+2x+5)^3}\,\mathrm{d}x$

$$= \frac{3}{256}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{3}{128}\left(\frac{x+1}{x^2+2x+5}\right) + \frac{1}{16}\left(\frac{x-7}{(x^2+2x+5)^2}\right).$$

Example 23: Find $\int \frac{dx}{\sin x . \cos x}$.

Solution: Let $u = \sin x$. Then $du = \cos x\, dx$.

Then $\int \frac{dx}{\sin x . \cos x} = \int \frac{du}{\sin x . \cos^2 x}$

$$= \int \frac{du}{u(1-u^2)}.$$

$$= \int \left(\frac{1}{u} + \frac{½}{1-u} - \frac{½}{1+u} \right) du$$

$$= \log u + ½\log(1-u) - ½\log(1+u) + c$$

$$= \log u\sqrt{\frac{1-u}{1+u}} + c$$

$$= \log\left(\sin x . \sqrt{\frac{1-\sin x}{1+\sin x}}\right) + c$$

Example 24: Find a reduction formula for $\int \frac{x^n}{1+x^2}\, dx$ and use it to find I_5 and I_6.

Solution: Let $I_n = \int \frac{x^n}{1+x^2}\, dx$.

$$= \int \frac{x^{n-2}(1+x^2-1)}{1+x^2}\, dx$$

$$= \int x^{n-2}\,dx - \int \frac{x^{n-2}}{1+x^2}\,dx$$

$$= \frac{1}{n-1}\,x^{n-1} - I_{n-2}.\ \text{if } n \geq 2$$

$$I_0 = \int \frac{1}{1+x^2}\,dx \;= \tan^{-1} x + c$$

$$I_1 = \int \frac{x}{1+x^2}\,dx \;= \frac{1}{2}\int \frac{du}{u}\ (\text{where } u = 1 + x^2)$$

$$= \tfrac{1}{2} \log u$$

$$= \tfrac{1}{2} \log(1 + x^2).$$

$I_2 = x - I_0 = x - \tan^{-1} x + c.$

$I_3 = \tfrac{1}{2} x^2 - I_1 = \tfrac{1}{2} x^2 - \tfrac{1}{2} \log(1 + x^2) + c.$

$$I_4 = \frac{1}{3}x^3 - I_2 = \frac{1}{3}x^3 - (x - \tan^{-1}x) + c$$

$$= \frac{1}{3}x^3 - x + \tan^{-1}x + c.$$

$$I_5 = \frac{1}{4}x^4 - I_3$$

$$= \tfrac{1}{4} x^4 - (\tfrac{1}{2} x^2 - \tfrac{1}{2} \log(1 + x^2)) + c$$

$$= \tfrac{1}{4} x^4 - \tfrac{1}{2} x^2 + \tfrac{1}{2} \log(1 + x^2) + c$$

$$I_6 = \frac{1}{5}x^5 - I_4 = \frac{1}{5}x^5 - \frac{1}{3}x^3 + x - \tan^{-1}x + c.$$

Note the quite different looking answers for *n* odd and *n* even.

§3.7. The t-Method

Let $t = \tan(x/2)$. Then the three basic trig functions can be expressed in terms of t:

$$\sin x = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-t^2}{1+t^2} \text{ and}$$

$$\tan x = \frac{2t}{1-t^2}.$$

Moreover $\dfrac{dt}{dx} = \tfrac{1}{2}\sec^2 t = \tfrac{1}{2}(1+t^2)$, so $dx = \dfrac{2dt}{1+t^2}$

So an integral that involves trig functions can be expressed as a rational function of t.

Example 25: Find $\displaystyle\int \frac{dx}{\cos x - 2\sin x + 2}$.

Solution: Let $t = \tan(x/2)$. Then $dx = \dfrac{2dt}{1+t^2}$.

Hence the integral is $\displaystyle\int \left(\frac{1}{\frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2} + 2}\right)\frac{2dt}{1+t^2}$

$$= \int \frac{2dt}{t^2 - 4t + 3} = \int \frac{2}{(t-1)(t-3)}\,dt.$$

Let $\dfrac{1}{(t-1)(t-3)} = \dfrac{A}{t-1} + \dfrac{B}{t-3}$.

Then $1 = A(t - 3) + B(t - 1)$ for all t.
Put $t = 1$. Then $A = -\frac{1}{2}$.
Put $t = 5$. Then $B = \frac{1}{2}$.

Hence $$\int \frac{2}{(t-1)(t-3)}\,\mathrm{dt} = \int \frac{1}{t-3}\,\mathrm{d}t - \int \frac{1}{t-1}\,\mathrm{d}t$$
$$= \log\left(\frac{t-3}{t-1}\right) + c$$
$$= \log\left(\frac{\tan(x/2) - 3}{\tan(x/2) - 1}\right) + c$$

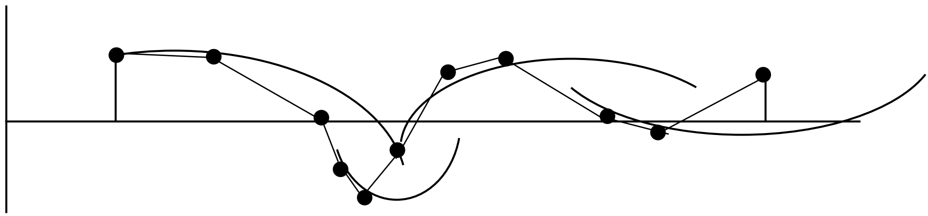
§3.8. Length of Curves

I would have dealt with lengths of curves, except for the fact that most of the integrals that arise need the techniques of integration from this chapter.

The length of a finite portion of a straight line is easy to compute. We just locate the endpoints (x_1, y_1) and (x_2, y_2) and calculate the distance between them:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

With more complicated curves we can approximate the curve by a series of straight lines.

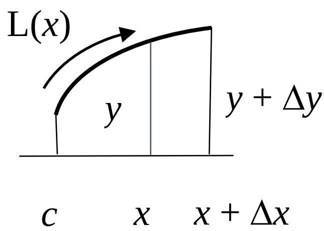


The estimate is the sum of the lengths of the line segments and clearly the more lines we use the more accurate we can make the estimate of the length. Take $x = c$ as a reference point and define L(x) as the length of the curve from c to x.

Theorem 4: $L(x) = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Proof: L(x) is the length of the curve from some unspecified starting point.

Let Δx and Δy be corresponding increments in x and y.



Hence $L(x + \Delta x) \approx L(x) + \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and so

$$\frac{L(x + \Delta x) - L(x)}{\Delta x} \approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ .$$

In the limit this approximation becomes exact, $\dfrac{\Delta y}{\Delta x} \to \dfrac{dy}{dx}$ and the LHS approaches $\dfrac{dL(x)}{dx}$.

$$\text{Hence } \frac{dL(x)}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ .$$

Using the Fundamental Theorem of Calculus:

$$L(x) = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ .$$

Example 26: Use integration to find the circumference of a circle with radius r.

Solution: The equation of the circle in the first quadrant is $y = \sqrt{r^2 - x^2}$ and $\dfrac{dy}{dx} = \dfrac{-2x}{2\sqrt{r^2 - x^2}} = \dfrac{-x}{\sqrt{r^2 - x^2}}$.

So $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{r^2 - x^2 + x^2}{r^2 - x^2} = \dfrac{r^2}{r^2 - x^2}$

The circumference $= 4r \displaystyle\int_0^r \frac{1}{\sqrt{r^2 - x^2}}\, dx$.

Let $x = r \sin \theta$.

Then $\dfrac{dx}{d\theta} = r \cos \theta$ and so $\dfrac{d\theta}{dx} = \dfrac{1}{r \cos \theta}$.

In the context of an integral we may write dx = r cos θ dθ.

Moreover $\sqrt{r^2 - x^2}$ = r cos θ and θ = π/2 when x = r.

So the circumference $= 4r \int_0^r \frac{1}{\sqrt{r^2 - x^2}}\, dx$.

$$= 4r \int_0^{\pi/2} \frac{1}{r \cos \theta}\, r \cos \theta \, d\theta$$

$$= 4r \int_0^{\pi/2} d\theta$$

$$= 4r \left[\theta\right]_0^{\pi/2}$$

$$= 4r \frac{\pi}{2} = 2\pi r.$$

Example 27: Find the length of the curve y = log cos x from x = 0 to x = π/3.

Solution: $\frac{dy}{dx} = \frac{1}{\cos x} .(-\sin x) = -\tan x.$

$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x.$

The length = $\displaystyle\int_0^{\pi/3} \frac{1}{\cos x}\, dx$.

Let $t = \tan(x/2)$. Then $dx = \dfrac{2}{1 + t^2}$ and $\cos x = \dfrac{1 - t^2}{1 + t^2}$.

$$\therefore \text{ the length} = \int_0^{1/\sqrt{3}} \left(\frac{1 + t^2}{1 - t^2}\right)\left(\frac{2}{1 + t^2}\right) dt$$

$$= \int_0^{1/\sqrt{3}} \left(\frac{2}{1 - t^2}\right) dt$$

$$= \int_0^{1/\sqrt{3}} \left(\frac{1}{1 - t} + \frac{1}{1 + t}\right) dt$$

$$= \Big[-\log(1 - t) + \log(1 + t)\Big]_0^{1/\sqrt{3}}$$

$$= \left[\log \frac{1 + t}{1 - t} \right]_0^{1/\sqrt{3}}$$

$$= \log \left(\frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}} \right) - \log 1$$

$$= \log \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right).$$

Example 28: Find the length of the curve $y = \log x$ from $x = 1$ to $x = \sqrt{3}$.

Solution: $\frac{dy}{dx} = \frac{1}{x}$, so $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x^2}$.

So the length is $\int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^{\sqrt{3}} \sqrt{\frac{x^2 + 1}{x^2}}\, dx$.

Let $x = \tan\theta$. Then $\frac{dx}{d\theta} = \sec^2\theta$.

The length = $\int_{\pi/4}^{\pi/3} \sqrt{\frac{\tan^2\theta + 1}{\tan^2\theta}} \sec^2\theta \, d\theta$

$$= \int_{\pi/4}^{\pi/3} \sqrt{\frac{\sec^2\theta}{\tan^2\theta}} \sec^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan\theta} \sec^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sin\theta \cos^2\theta} \, d\theta$$

Let $u = \cos\theta$. Then $\dfrac{du}{d\theta} = -\sin\theta$.

Hence the length $= \displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{1}{\sin^2\theta \cos^2\theta} \, du$

$$= \int_{1/2}^{1/\sqrt{2}} \frac{1}{u^2(1-u^2)} \, du \, .$$

Let $\dfrac{1}{u^2(1-u^2)} \equiv \dfrac{A}{u^2} + \dfrac{B}{u} + \dfrac{C}{1-u} + \dfrac{D}{1+u}$

$$\equiv \frac{A(1-u^2) + Bu(1-u^2) + Cu^2(1+u) + Du^2(1-u)}{u^2(1-u^2)}$$

$$\equiv \frac{(-B + C - D)u^3 + (-A + C + D)u^2 + Bu + A}{u^2(1-u^2)}\ .$$

Hence A = 1, B = 0, –B + C – D = 0 and –A + C + D = 0.

Thus A = 1, B = 0, C = D = ½ .

So the length = $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{1}{u^2}\,du + \frac{1}{2}\int_{1/2}^{1/\sqrt{2}} \left[\frac{1}{1-u} + \frac{1}{1+u}\right] du$

$$= \left[\frac{1}{-3u^2}\right]_{1/2}^{1/\sqrt{2}} + \frac{1}{2}\left[\log\frac{1+u}{1-u}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{12} - \frac{1}{2}\log 3 + \frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \log\sqrt{\frac{\sqrt{2}+1}{\sqrt{6}-\sqrt{3}}} - \frac{1}{12}\ .$$

§3.9. Parametric Representation of Curves

We are used to a curve in $\mathbb{R}^2$ being described by a function $y = f(x)$. That's OK for a parabola, but for more complicated curves it is unsuitable, because such a function gives one y-value for every x-value. In the case of a circle we would have to consider only a hemisphere.

A more suitable way of describing a curve is by means of a parameter. A curve in $\mathbb{R}^2$ can be described as a pair of real functions, $\varphi(t) = (x(t), y(t))$ and a curve in $\mathbb{R}^2$ can be described as a triple of real functions, $\varphi(t) = (x(t), y(t), z(t))$. Using parameters we can describe the most complicated of curves that wind around like a scribble on paper.

The other advantage of using parameters is that we get a sense of direction along the curve. As t increases the curve is traced out in a certain direction.

In fact we also get a notion of speed. We can think of t as time and the position of a point traversing the curve is given by $(x(t), y(t))$.

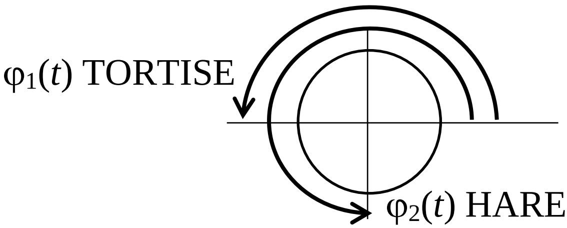
Example 29: The functions $\varphi_1 = (2\cos 2\pi t, 2\sin 2\pi t)$ describes a circle with centre (0, 0) and radius 2. In actual fact it describes a point moving around that circle at uniform speed.

As t goes from 0 to 1 the point starts at (2, 0) and moves anti-clockwise around the circle before returning to (2, 0).

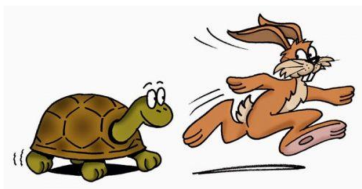
The function:

$\varphi_2(t) = (2\cos 2\pi(1 - t^2), 2\sin 2\pi(1 - t^2))$

also describes a point moving around the same circle, again starting and finishing at (2, 0). But, if we think of t as representing time in minutes, the point starts off quite fast, reaching (2 cos 3π/2, 2 sin 3π/2) = (0, – 2) after 30 seconds (t = 1/2) while the first point will only have reached (– 2, 0) in that time.



Think of $\varphi_1(t)$ as representing the tortoise and φ_2(t) as the hare, in Aesop's fable. But, like the hare the second point goes very fast at the beginning, but slows down, while the first point goes slow and steady and they both reach the same endpoint at the same time.



The parametric version is particularly useful for arc length. If L(t) is the length of the curve from some unspecified starting point then ΔL approximates the distance between the point (x, y) and $(x + \Delta x, y + \Delta y)$.

So $\Delta L \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Hence $\dfrac{\Delta L}{\Delta t} \approx \sqrt{\left(\dfrac{\Delta x}{\Delta t}\right)^2 + \left(\dfrac{\Delta y}{\Delta t}\right)^2}$.

As $\Delta t \to 0$, these fractions approach the respective derivatives and the approximation becomes exact.

So $\dfrac{dL}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ and hence

$$L(t) = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \,.$$

If we want the length of the curve between $t = a$ and $t = b$, it is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \,.$$

Example 30: Find the length of the curve $\varphi(t) = (r \cos t, r \sin t)$ from $t = 0$ to $t = \theta$.

Solution: $x = r \cos t$ so $\dfrac{dx}{dt} = -\, r \sin \theta$.

$$y = r\sin t \text{ so } \frac{dy}{dt} = r\cos t.$$

$$\therefore\ L = \int_0^\theta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = r\int_0^\theta \sqrt{\left((-\sin t)^2 + \cos^2 t\right)}\, dt$$

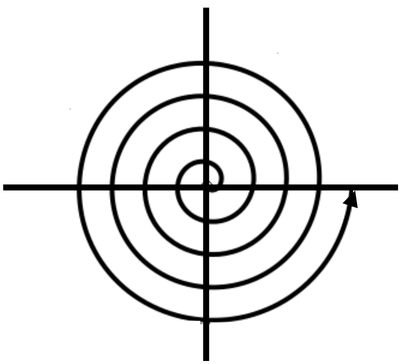
$= r\int_0^\theta dt = r\theta$. But we knew that already. The length of an arc of radius r, subtending an angle θ at the centre (in radians of course) is rθ.

Example 31: Let a curve be described by:

$$\varphi(t) = (t\cos 2\pi t, t\sin 2\pi t).$$

Find the length, L, of the curve from $t = 0$ to $t = 4$.

Solution: Let $x = t\cos 2\pi t$ and $y = t\sin 2\pi t$. Then, as t increases from 0 to 4, (x, y) starts at (0, 0) and moves out in a spiral. It rotates 4 times anticlockwise until it reaches (4, 0).



Now $\frac{dx}{dt} = \cos 2\pi t - 2\pi t \sin 2\pi t$ and

$\frac{dy}{dt} = \sin 2\pi t - 2\pi t \cos 2\pi t$.

Hence L is

$$\int_0^4 \sqrt{(\cos 2\pi t - 2\pi t \sin 2\pi t)^2 + (\sin 2\pi t - 2\pi t \cos t)^2}\, dt$$

$$= \int_0^4 \sqrt{1 + 4\pi^2 t^2}\, dt$$

At this stage we could use numerical techniques. Using Simpson's Rule with 4 strips we get the estimate of 50.7727.

However, now that we have developed some techniques of integration it would be instructive to evaluate the integral so as to find the exact value.

Whenever we have an expression $a^2 + x^2$ we should consider the substitution $x = a \tan \theta$. This is because the derivative of $\tan \theta$ is $\sec^2\theta = 1 + \tan^2\theta$.

So here, let $t = \frac{1}{2\pi} \tan \theta$.

Then $dt = \frac{1}{2\pi}\sec^2\theta\, d\theta$ and $\sqrt{1 + 4\pi^2 t^2} = \sec\theta$.

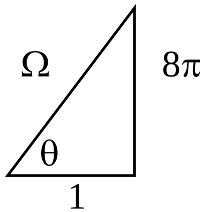
$$\therefore\ L = \frac{1}{2\pi}\int_0^{\tan^{-1}(8\pi)} \sec^3\theta\, d\theta = \frac{1}{2\pi}\int_0^{\tan^{-1}(8\pi)} \frac{1}{\cos^3\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{\tan^{-1}(8\pi)} \frac{1}{\cos\theta\,(1 - \sin^2\theta)}\, d\theta\,.$$

Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$.

When $\theta = \tan^{-1}(8\pi)$, $\tan\theta = 8\pi$ and so and

$\sin\theta = \dfrac{8\pi}{\sqrt{1 + 64\pi^2}}$. Let's denote this by Ω.



Or, you could write: $64\pi^2 = \tan^2\theta = \sec^2\theta - 1$.

So $\cos\theta = \dfrac{1}{\sqrt{\sec^2\theta}} = \dfrac{1}{\sqrt{1 + 64\pi^2}}$.

I'm sure you'll agree that arguing from the diagram is much easier. Whenever you have one trig value and you want another for the same angle, the little right-angled triangle is by far the easier way.

We have $L = \frac{1}{2\pi}\int_0^{\Omega} \frac{1}{(1-u^2)^2}\,du$.

We now use Partial Fractions.

$$\frac{1}{(1-u^2)^2} \equiv \frac{1}{(1-u)^2(1+u)^2}$$

$$\equiv \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{(1+u)^2} + \frac{D}{1+u}$$ for some constants A, B, C, D.

When I first learnt Partial Fractions I was taught to use '≡' rather than '=' to emphasise that the two expressions are equal for all values of u, rather than this being an equation to solve. There's probably no good reason for doing this, but old habits die hard.

Hence $1 \equiv A(1+u)^2 + B(1-u)(1+u)^2 + C(1-u)^2 + D(1+u)(1-u)^2$.

There are two techniques for finding these constants. One is to substitute certain values of u which make for simpler equations.

Let $u = 1$. Then 4A = 1 and so A = ¼ .

Let $u = -1$. Then 4C = 1 and so C = ¼ .

We could substitute any other value, but let's use the other technique of equating corresponding coefficients.

The constant terms (the same as substituting $u = 0$) give us A + B + C + D = 1, which reduces to B + D = ½ . The coefficients of u^3 give us – B + D = 0, so:
A = B = C = D = ¼ .

We can now write:

$$L = \frac{1}{8\pi}\int_0^{\Omega}\frac{1}{(1-u)^2}\,du + \frac{1}{8\pi}\int_0^{\Omega}\frac{1}{(1+u)^2}\,du$$

$$+ \frac{1}{8\pi}\int_0^{\Omega}\frac{1}{(1-u)}\,du + \frac{1}{8\pi}\int_0^{\Omega}\frac{1}{(1+u)}\,du$$

$$= \frac{1}{8\pi}\left[\frac{1}{1-u}\right]_0^{\Omega} - \frac{1}{8\pi}\left[\frac{1}{1+u}\right]_0^{\Omega}$$

$$+ \frac{1}{8\pi}\left[-\log(1-u)\right]_0^{\Omega} + \frac{1}{8\pi}\left[\log(1+u)\right]_0^{\Omega}$$

After a fair amount of simplification this becomes:

$$L = 2\sqrt{1 + 64\pi^2} + \frac{1}{4\pi}\log\left(\sqrt{1 + 64\pi^2} + 8\pi\right)$$

$$\approx 50.6171$$

Now we could have obtained a fairly accurate estimate with almost no work by supposing that the average radius on the first circuit is 0.5, on the second, 1.5, and so on. Our rough and ready estimate would be:

$2\pi(0.5 + 1.5 + 2,5 + 3.5) = 16\pi = 50.2655$.

But in certain cases there's an even easier way.

Example 32: A roll of newsprint has an external diameter of 1 metre and an internal diameter of 20cm. If the thickness of the paper is 0.1mm what is the length of the paper on the roll?

Solution: Although the paper forms a spiral, we can assume that it is made up of concentric circles, and proceed as above. But an even easier way to get a good estimate is to imagine the paper unrolled, If the length in metres is L, the area of long edge of the paper is:

0.0001L square metres.

But the paper part of the end of the roll is an annulus with an outside radius of 0.5 metres and an inside radius of 0.1 metres. The area of this annulus is:

$\pi(0.5^2 - 0.1^2) = 0.24\pi = 0.754$ m^2.

Hence 0.0001L = 0.754 and so L = 7540 metres.

The moral of the story is that you don't always need to work out an exact answer to a practical problem. The skill in Applied Mathematics is knowing how to get the right balance between simplicity and accuracy.

§3.10. Integral Closed Spaces

This final section in the chapter is of only mild academic interest. The main reason for including it is to give you plenty of practice with the techniques of integration. But first, let's consider differentiation.

Differentiable functions on ℝ form a vector space over ℝ, because the sum and difference of two differentiable functions is differentiable and any constant multiple of a differentiable function is differentiable.

A **differentiable closed space** is a subspace that contains the constant functions and is closed under differentiation. That is, the derivative of every function in V is also in V.

The smallest differentiable closed space is clearly the space of all constant functions. So is the space of all linear functions $ax + b$, together with the constant functions. The space of all quadratics $ax^2 + bx + c$ (including the degenerate quadratics that are really linear functions or constant functions) is a differentiable closed space of dimension 3. A slightly more interesting example is the space of all polynomials.

The set of all functions of the form a sin x is not differentiable closed, but $\{a \sin x + b \cos x \mid a, b \in \mathbb{R}\}$ is.

Of slightly more interest are integrable closed functions. Integrable functions on $\mathbb{R}$ form a vector space over $\mathbb{R}$. An **integral closed space** is a subspace, V, that contains the constant functions and is closed under integration. That is, the integral of every function in V is also in V.

The reason for insisting on an integral closed space to contain the constant functions is because indefinite integrals contain arbitrary constants. So the space of all real multiples of e^x is not integral closed because the integral of e^x is not e^x, but $e^x + c$.

A simple example of an integral closed space is the space of all polynomials, which is infinite-dimensional over $\mathbb{R}$. In fact, because every integral closed space must contain the constant functions, they must contain all their integrals, and integrals of integrals, and so on. So every integral closed space must contain all real polynomials. While there are many finite-dimensional differentiable closed spaces, every integral closed space must contain all real polynomials and so must be infinite dimensional over $\mathbb{R}$.

Example 33: The space of all real polynomials of the form a(x) + k e^x where a(x) is a real polynomial and $k \in \mathbb{R}$ is integral closed.

Example 34: The space of all real polynomials of the form: a(x) + k sin x + h cos x where a(x) is a real polynomial and h, $k \in \mathbb{R}$, is integral closed.

Theorem 6: The space of all real polynomials of the form a(x) + b(x) sin x + c(x) cos x. where a(x), b(x), c(x) and d(x) $\in \mathbb{R}]x]$, is integral closed.

Proof: Let V be the set of such polynomials.

It is sufficient to check that $\int x^n \sin x \, dx$ and $\int x^n \cos x \, dx$ are both in V for all n.

Let $I_n = \int x^n \sin x \, dx$ and $J_n = \int x^n \cos x \, dx$.

We prove by induction on n that both I_n and J_n are in V.

It is true for n = 0.

Suppose it is true for n.

Consider $\int x^{n+1} \sin x \, dx$.

Let $u = x^{n+1}$ and $dv = \sin x$ dx.

Then $du = (n + 1)x^n \, dx$ and $v = - \cos x$.

Hence $I_{n+1} = -x^{n+1}\cos x + (n+1)\int x^n \cos x\,dx$

$$= -x^{n+1}\cos x + (n+1)J_n \in V \text{ by induction.}$$

Consider $\int x^{n+1}\cos x\,dx$.

Let $u = x^{n+1}$ and $dv = \cos x\,dx$.

Then $du = (n+1)x^n\,dx$ and $v = \sin x$.

Hence $J_{n+1} = x^{n+1}\sin x - (n+1)\int x^n \sin x\,dx$

$$= -x^{n+1}\sin x - (n+1)I_n \in V \text{ by induction.}$$

It is not particularly surprising that sin x and cos x are intertwined in such a way. More surprising is the close relationship between the functions $\tan^{-1} x$ and $\log(1 + x^2)$. The connection is that both give similar rational functions when differentiated. The derivative of $\tan^{-1}x$ is $\frac{1}{1+x^2}$ and the derivative of $\log(1 + x^2)$ is $\frac{2x}{1+x^2}$.

Theorem 7: The space, V, of all real polynomials of the form:

$$a(x) + b(x)\tan^{-1}x + c(x)\log(1 + x^2) + \frac{d(x)}{1+x^2},$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are real polynomials, is integral closed.

Proof: Clearly $\int x^n \, dx \in V$ for all n and hence $a(x) \in V$ for all $a(x) \in \mathbb{R}[x]$.

Let $I_n = \displaystyle\int \frac{x^n \, dx}{1 + x^2}$,

$J_n = \int x^n \tan^{-1} x \, dx$ and

$K_n = \int x^n \log(1 + x^2) \, dx$.

It's sufficient to prove that for all integers $n \geq 0$, all three of I_n, J_n and K_n are in V.

$I_n = \displaystyle\int \frac{x^n}{1 + x^2} \, dx$:

$$I_n = \int \frac{x^n}{1 + x^2} \, dx = \int \frac{x^{n-2}(1 + x^2 - 1)}{1 + x^2} dx$$

$$= \int x^{n-2} \, dx - \int \frac{x^{n-2}}{1 + x^2} \, dx$$

$$= \frac{1}{n-1} x^{n-1} - I_{n-2}. \text{ if } n \geq 2$$

Now $I_0 = \displaystyle\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + c$ and

$I_1 = \displaystyle\int \frac{x}{1 + x^2} \, dx = ½ \log(1 + x^2)$.

Clearly both are in V.

Since $I_n = \frac{1}{n-1} x^{n-1} - I_{n-2}$. if $n \geq 2$ we can prove, by induction, that $I_n \in V$ for all $n \geq 0$.

$J_n = \int x^n \tan^{-1} x$:

Let $u = \tan^{-1}x$ and $dv = x^n\, dx$.

Then $du = \frac{1}{1+x^2}\, dx$ and $v = \frac{1}{n+1} x^{n+1}$

Hence $J_n = \frac{1}{n+1} x^{n+1} \tan^{-1}x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2}\, dx$

$$= \frac{1}{n+1} x^{n+1} \tan^{-1}x - \frac{1}{n+1} I_{n+1} \in V.$$

$K_n = = \int x^n \log(1 + x^2)\, dx$:

Let $u = \log(1 + x^2)$ and $dv = x^n$.

Then $du = \frac{2x}{1+x^2}\, dx$ and $v = \frac{1}{n+1} x^{n+1}$.

So $K_n = \frac{1}{n+1} x^{n+1} \log(1 + x^2) - \frac{2}{n+1} \int \frac{x^{n+2}}{1+x^2}\, dx$

$$= \frac{1}{n+1} x^{n+1} \log(1 + x^2) - \frac{2}{n+1} I_{n+2} \in V.$$

Hence V is integral closed. ✋☺